MAC-CPTM Situations Project

Situation 41: Square Roots

Prompt

A teacher asked her students to sketch the graph of $f(x)=\sqrt{-x}$. A student responded, "That's impossible! You can't take the square root of a negative number!"

Commentary

This situation addresses several key concepts that occur frequently in school mathematics: additive inverse, negative numbers, function, domain, and range. Since the symbol '-' has multiple interpretations, it is important to distinguish between a negative number and the additive inverse (i.e. opposite) of a number. Moreover, the domain over which a function is defined determines the range of the function, and a table of values provides an example to illustrate the relationship between domain and range. For a set of points with coordinates $(x, f(x))$ to define the graph of a function, each first coordinate, x, must correspond to a unique second coordinate, $f(x)$. A graphical representation highlights the univalence of f as a function of x.

Mathematical Foci

Mathematical Focus 1

The symbol '-' is used to refer to more than one mathematical idea. Two of these ideas that are commonly confused are 'negative' and 'additive inverse' (also called 'opposite').

Mathematical terms have precise meaning. The symbol "-" is commonly read as both negative and opposite. However, a negative number is a *kind* of number, while the opposite of a number describes a *relationship* of one number of another.

A negative real number has a value less than zero, while the additive inverse of a real number is the value such that the sum of the number and its inverse is the additive identity, zero. For example, "negative 6" indicates a number less than zero, and "the number opposite of positive 6" indicates the additive inverse of +6, which is -6. Every real number has an additive inverse, and only in the case of 0 is a number its own additive inverse.

In the example above, -6 and the additive inverse of 6 are the same, but this is not always the case. Take, for example, clock arithmetic. Consider a 24 hour clock in which 24:00 is the same as 0:00. One could ask, “What is the additive inverse of 11:00?” It is not -11:00 because there is no such time. Rather, the additive inverse is 13:00 because 11:00+13:00 = 24:00 = 0:00.

Using the variable x to represent a number does not indicate whether the number is positive, negative, or zero. Since $-x$ represents the additive inverse of x, and not necessarily a negative value, it could be positive, negative, or zero, depending on the value of x.

Mathematical Focus 2

The domain of a function is critical in determining the values of the range of the function.

The implicit assumption that the domain and range of a function are restricted to real numbers could contribute to the popularity of the statement "You can't take the square root of a negative number." If one assumes that the range of the function *f* with the rule $f(x)=\sqrt{-x}$ is in the real numbers, then the function's domain must be in $x \leq 0$. Similarly, if one assumes that the range of the function *h* with rule $h(x)=\sqrt{x+2}$ is in the real numbers, then the domain must be in $x \geq -2$. If the domain of the function *f* with rule $f(x)=\sqrt{-x}$ includes all real numbers, then the range of *f* is in the set of complex numbers.

The table below provides an example that illustrates how the domain of the function *f* with rule $f(x)=\sqrt{-x}$ determines values over which the range of *f* is defined:

x	$\sqrt{-x}$
-4	$\sqrt{-(-4)}=2$
-3	$\sqrt{-(-3)}=\sqrt{3}$
-2	$\sqrt{-(-2)}=\sqrt{2}$
-1	$\sqrt{-(-1)}=1$
0	$\sqrt{-0}=0$
1	$\sqrt{-1}=i$
2	$\sqrt{-2}=i\sqrt{2}$
3	$\sqrt{-3}=i\sqrt{3}$
4	$\sqrt{-4}=2i$

The values in the table above are consistent with the ordered pairs for the function *f* with rule $f(x)=\sqrt{-x}$. If $x \leq 0$, the radicand ≥ 0, and the range of *f* is the positive real numbers, including zero. If $x > 0$, the radicand < 0, and the range of *f* is in the set of complex numbers.

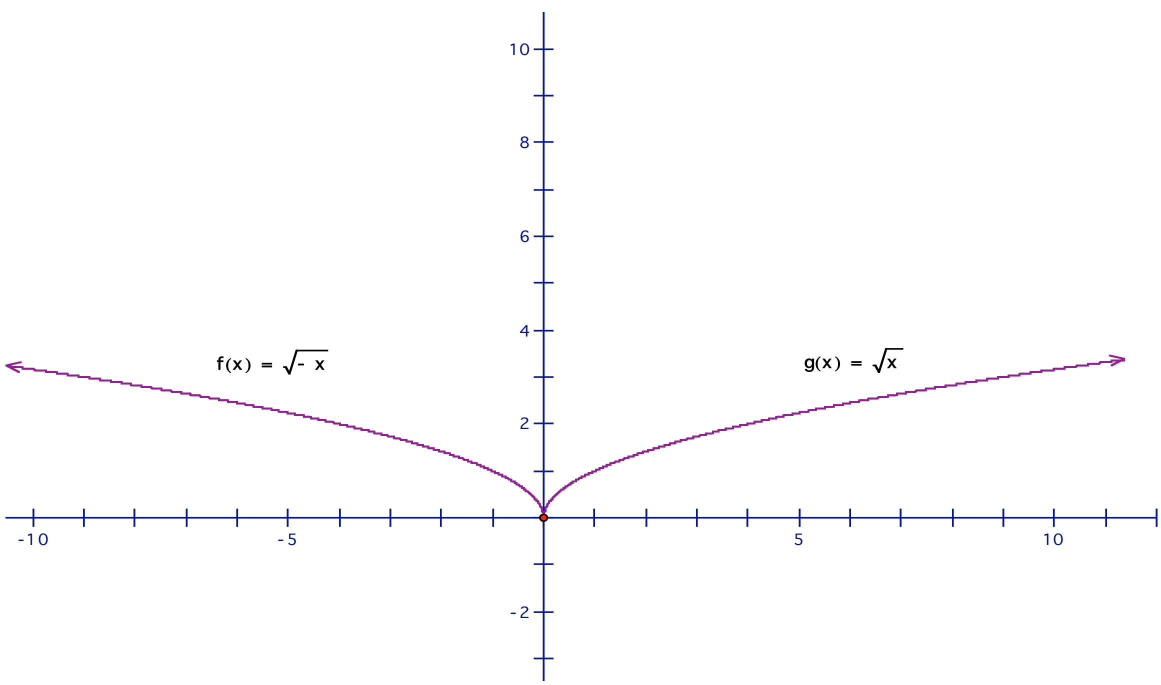
Mathematical Focus 3

The interpretation of $-x$ *as the additive inverse of* x *has implications in the graphs of functions. Specifically, the graph of the points* $(-x, g(x))$ *is a reflection across the vertical axis of the graph of the points* $(x, g(x))$*.*

In the real numbers, the additive inverse of a number is the same as the negative of the number (see Focus 1). Consider the real number line. The positive and negative numbers are, in a sense, reflections of each other across 0. That is, a number and its additive inverse are on opposite sides of 0 on the number line, and also the same distance from 0.

In a similar way, one can understand values of x in a coordinate plane as being on the opposite side of the vertical axis as the values of $-x$, and the same distance from the vertical axis. So the point $(x, g(x))$ is a reflection across the vertical axis of the point $(-x, g(x))$. Applying this to the current Situation, there does exist a function defined by the rule $f(x)=\sqrt{-x}$ (for domain $x \le 0$), and it is a reflection across the vertical axis of the function defined by the rule $g(x)=\sqrt{x}$ (with domain $x \ge 0$).

Graphs of the functions f and g are shown below.



Post-Commentary

The concept of function involves a pairing between two sets. This pairing is often given as a rule (or rule of assignment), such as $f(x) = x + 3$. The terminology involved is far from consistent, so an explanation is offered here.

Typically, the domain of a function refers to the elements that the function "acts on" or "maps from." In college mathematics, the domain is often prescribed a priori any rule of assignment. Thus, the notation $h: A \to B$ could be used to denote the function h with domain A, without explicitly saying how h maps from A to B. In secondary mathematics, however, the rule of assignment is often established first, and the domain is assumed to be the maximal real domain that can be used with the rule of assignment. For example, a high-school algebra class might be given the rule $g(x) = \frac{1}{x}$ and then be expected to deduce that the domain is all real numbers except 0. In both cases, however, every element in the domain must be "mappable." In the example of the function g, it would be inappropriate in either case to state that g has domain $\mathbb{R}$, since 0 is an element of $\mathbb{R}$ and $\frac{1}{0}$ is not well-defined.

$\mathbb{R}$-{0}

The term range is likewise used in multiple ways. In high-school mathematics, the range is usually understood to mean the actual set of function values the rule of assignment yields when applied to the domain. Thus, the domain $\mathbb{R}$ - {0} and rule $x \mapsto \frac{1}{x}$ would result in a range of $\mathbb{R}$ - {0}. This usage of the word range results in all functions being onto (or surjective). In college mathematics, range will generally refer to a set that simply *contains* the function values when the rule of assignment is applied to the domain. Thus, there are many sets which could serve as the range for a function with domain $\mathbb{R}$ - {0} and rule $x \mapsto \frac{1}{x}$, including $\mathbb{R}$ - {0}, $\mathbb{R}$, and $\mathbb{C}$. Taken this way, the range will generally be specified by the one proposing the function, and choosing a different range results in a different function while another term, 'image,' will usually be used to describe the actual set of function values for a given rule of assignment and domain.

Finally, the rule of assignment of a function will most often be constrained. Most of the time, the rule of assignment for the function must be constructed so that each element of the domain maps to precisely one element of the range. Rarely, the word 'function' will be used to describe any pairing between the domain and range sets, but the term 'relation' is more common when this occurs. In these situations, we will abide by the high-school mathematics conventions for domain and range, and we will only use the term 'function' in the constrained sense unless explicitly stated otherwise.